Photomultiplier tube spectra and absolute calibration

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Outline

- ***In these set of slides we will derive the expected spectra for photo-multiplier tubes. The techniques can be applied to other detectors as well.***

- ***To derive these we will need some powerful mathematical tools. We will also use some basic understanding of the device.***

- ***The assumption in these slides is that we have integrated the charge coming from a PMT. The only sources of noise is the PMT itself.***

- ***The electrical signal out of a PMT has been analyzed in a separate, earlier two lectures.***

- ***Reference: also see E. Bellamy, et al., NIM A339, 468 (1994), and also the Hamamatsu, Photomultiplier tubes: Basics and Applications (2007) 3rd edition.***

Some definitions

X is a continuous random variable with probability density function $P_X(x)$ then the characteristic function is

$$\varphi_X(k) \equiv E[e^{ikx}] = \int_{-\infty}^{\infty} P_X(x) e^{ikx}\, dx$$

X is a random variable, x is a realization of X over its domain. $E[\ ]$ is the expectation value of its argument. What is the point of the characteristic function ?

It is a way of tagging the probability number with a unique number (e^{ikx}). It is as if we are storing the number in a file folder with a tag.

Notice $\varphi(k=0)=1$ since it is the integral of the PDF.

the mean is given by $\langle x \rangle = -i\dfrac{\partial\varphi}{\partial k}(k=0)$, and so on for other moments

If X and Y are two random variables and $z = f(x,y)$ then the Characteristic function for Z is

$$\varphi_Z(k) = \iint e^{ikf(x,y)} P_X(x) dx\, Q_Y(y) dy$$

To get the moments of $f(x,y)$ often it is not necessary to evaluate the integral.

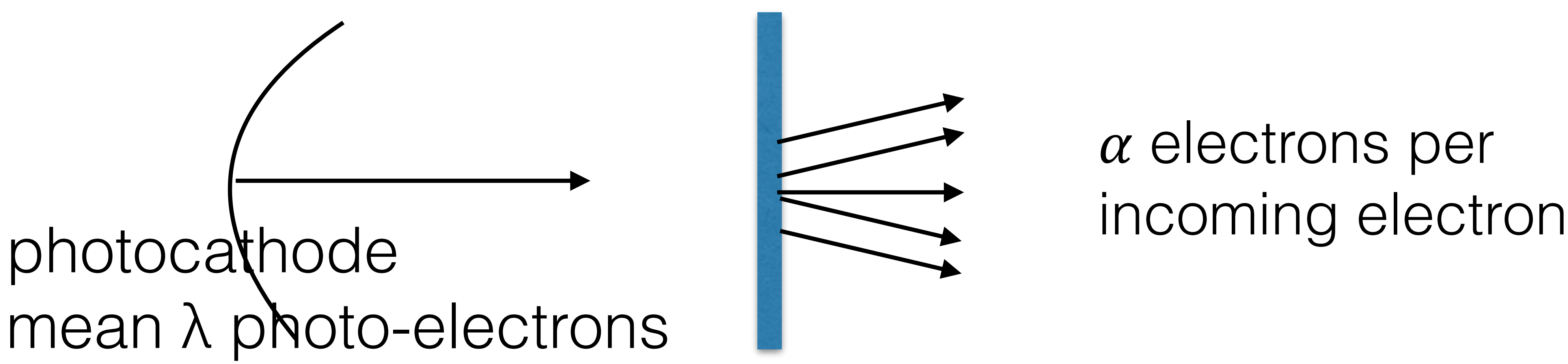
e.g. $f(x,y) = x + y \Rightarrow \varphi_Z(k) = \varphi_X(k).\varphi_Y(k)$ .... leave it for you to prove this

basics of photomultiplier

Mean of λ photo electrons come from the photo-cathode to the first dynode.

Each electron generates α electrons at the first dynode.

Each subsequent stage produces gain of few electrons per incoming electrons leading to gains of ~10^{6-7}



If mean of λ photons convert in a photo-sensor with a mean gain of α electrons per photon what is the distribution of the output number of electrons ?

Basically, an average of λ packets arrive each with an average of α items in each packet. What is the mean and variance of the total number of items ? How do we calculate this…

Distribution of incoming electrons with Poisson mean of λ

K is the number of electrons, a discrete random variable with probability mass function

$$\mathrm{P}_K(\mathrm{k}) = \frac{\lambda^k}{k!}e^{-\lambda}$$

Characteristic function for this is the expectation value of e^{isk}

$$\varphi_K(s) \equiv E[e^{isk}] = \sum_{k=0}^{\infty} e^{isk} P_K(k) = \sum_{k=0}^{\infty} \frac{e^{isk}\lambda^k}{k!}e^{-\lambda} = e^{e^{is}\lambda}e^{-\lambda} = e^{\lambda(e^{is}-1)}$$

This function has many interesting properties.

Similarly, the distribution of photons from each electron has probability

$\mathrm{P}_L(\ell)=\dfrac{\alpha^\ell}{\ell!}e^{-\alpha}$ and a similar characteristic function $\varphi_L(s)$.

We have labeled the two functions to distinguish them from each other.

Additionally recall that the mean for a Poisson distribution with parameter λ is

$E[k] = \lambda$ and the variance is also $E[k^2] - (E[k])^2 = \lambda$

Simple calculation first

Before we do the full calculation we will perform an intuitative calculation.
Obviously λ and α are Poisson parameters.   Total charge will be called Q.

$$\langle Q \rangle = \lambda \bullet \alpha$$

The fractional variance of Q will have contribution from the fluctuation of the incident number K and then the fluctuation in the secondary number $\sum_{i=1}^{K} L_i$ which is a sum of K random numbers each Poisson distributed with parameter α

$$\frac{Var[Q]}{\langle Q \rangle^2} = \frac{Var[K]}{\langle K \rangle^2} + \frac{1}{K} \times \frac{Var[L]}{\langle L \rangle^2} = \frac{1}{\lambda} + \frac{1}{\lambda} \bullet \frac{1}{\alpha}$$

$$Var[Q] = \lambda\alpha(\alpha + 1)$$

Suppose L is actually Normally distributed with parameters α (mean) and σ(standard deviation)

$$\frac{Var[Q]}{\langle Q \rangle^2} = \frac{Var[K]}{\langle K \rangle^2} + \frac{1}{K} \times \frac{Var[L]}{\langle L \rangle^2} = \frac{1}{\lambda} + \frac{1}{\lambda} \bullet \frac{\sigma^2}{\alpha^2}$$

$$Var[Q] = \lambda(\sigma^2 + \alpha^2)$$

Now calculate the PDF for the charge

Total charge is a discrete random number Z.

$$Z = \sum_{i=1}^{K} L_i$$

This is a sum of K random numbers, each is the count from an electron gain.

Now, it is obvious that the mean number of total electrons must be $\lambda \times \alpha$, where λ is the Poisson mean for the number of electrons and α is the Poisson mean for the gain or the number of electrons resulting from the multiplication of an electron.

However, the random number for the total number of photons is not a product of the two random numbers K (the number of electrons) and L (the number of photons).

The characteristic function for the number Z is

$$\varphi_Z(s) \equiv E[e^{isz}] = \sum_{z=0}^{\infty} e^{isz} P_Z(Z = z)$$

Here P_Z is unknown.

Characteristic function of Z

Start with the generating function for total number Z

(recall that K is the r.v. for electrons and L is the r.v. for gain on each electron)

$$\varphi_Z(s) \equiv \sum_{z=0}^{\infty} e^{isz} P_Z(Z=z) = E[e^{isz}] = \sum_{k=0}^{\infty} E[e^{is\sum_{i=1}^{K} L_i} \big| K=k] \bullet P_K(k)$$

$$= \sum_{k=0}^{\infty} E[e^{isl_1} e^{isl_2} \ldots e^{isl_K} \big| K=k] \bullet P_K(k)$$

This says that the total expectation for e^{isz} is the same as

the average of the conditional expectation for k electrons (averaged over the probability of obtaining k electrons). This is the law of total expectation.

Each random variable L_i is independent, and so each has the same char. func.

$$\varphi_Z(s) = \sum_{k=0}^{\infty} (\varphi_L(s))^k \bullet P_K(k)$$

This leads to a compact expression.

$$\varphi_Z(s) = \sum_{k=0}^{\infty} (e^{\alpha(e^{is}-1)})^k \bullet \frac{e^{-\lambda}\lambda^k}{k!} = e^{\lambda(e^{\alpha(e^{is}-1)}-1)}$$

expressions for Poisson and Normal gain

For Poisson gain

$$\varphi_L(s) = e^{\alpha(e^{is}-1)}$$

For gain with normal PDF. $N(\mu,\sigma^2)$

$$\varphi_L(s) = e^{is\mu} \bullet e^{-\frac{s^2\sigma^2}{2}}$$

Generally for Poisson PDF with $\alpha >> 1$ we can use Normal PDF with $\mu=\alpha$ and $\sigma^2 = \alpha$

For Poisson

$$\varphi_Z(s) = \sum_{k=0}^{\infty} (e^{\alpha(e^{is}-1)})^k \bullet \frac{e^{-\lambda}\lambda^k}{k!} = \sum_{k=0}^{\infty} e^{k\alpha(e^{is}-1)} \bullet \frac{e^{-\lambda}\lambda^k}{k!}$$

When we invert this to get the probability we get a formula with an infinite series

$$P_Z(n) = \begin{cases} e^{-\lambda} + \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} \bullet e^{-k\alpha} & \text{for } n = 0 \\ \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} \bullet \frac{e^{-k\alpha}(k\alpha)^n}{n!} & \text{for } n > 0 \end{cases}$$

It is important to be careful about 0

Normal distributed gain

For Normal

$$\varphi_Z(s) = \sum_{k=0}^{\infty} (e^{is\mu} \bullet e^{-\frac{s^2\sigma^2}{2}})^k \bullet \frac{e^{-\lambda}\lambda^k}{k!} = \sum_{k=0}^{\infty} (e^{is\mu k} \bullet e^{-\frac{s^2\sigma^2 k}{2}}) \bullet \frac{e^{-\lambda}\lambda^k}{k!} = e^{\lambda\left[e^{is\mu - s^2\sigma^2/2} - 1\right]}$$

When we invert this to get the probability we get a formula with an infinite series

$$P_Z(z) = \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} \bullet \frac{1}{\sqrt{2\pi\sigma^2 k}} e^{-\frac{(z-k\mu)^2}{2\sigma^2 k}} \quad \text{for } z > 0$$

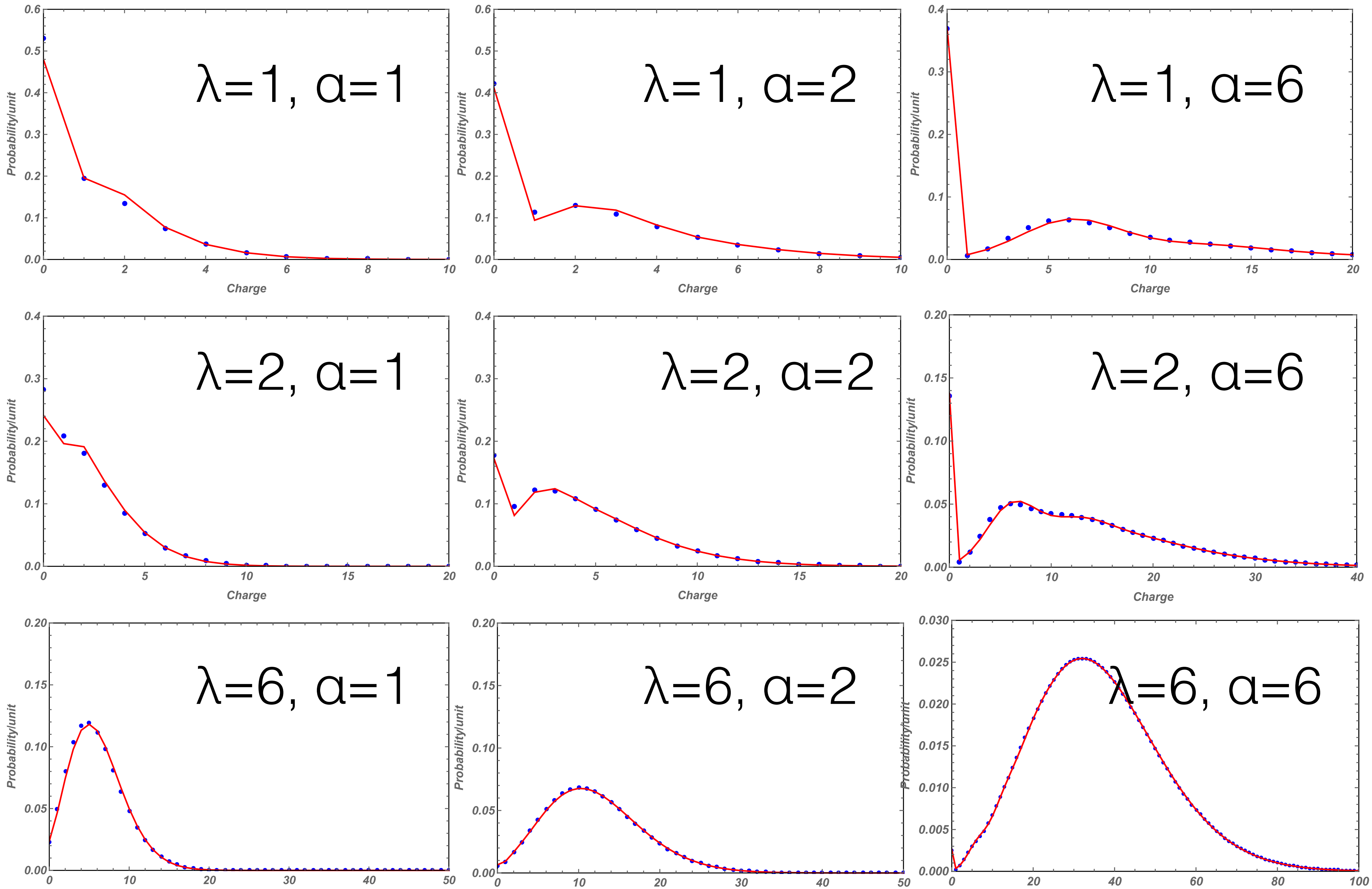
There is some care needed here because z is now a continuous variable, and $P_Z(z)$ is now a probability density function.

(When the bin size of z is chosen to be $\delta z = 1,$ we recover an approximation to the probability mass function for a Poisson jumping distribution.)

One still needs to obtain a well defined probability at $z = 0.$ For this we go back to setting μ=α and $\sigma^2 = \alpha$

$$P_Z(n) \approx \begin{cases} e^{-\lambda} + \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} \bullet \frac{e^{-k\alpha/2}}{\sqrt{2\pi k\alpha}} & \text{for } n = 0 \\ \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} \bullet \frac{1}{\sqrt{2\pi k\alpha}} e^{-\frac{(n-k\alpha)^2}{2k\alpha}} & \text{for } n > 0 \end{cases}$$

comparison of plots



Blue dots are for Poisson/Poisson, Red curve is Poisson/Normal approximation
λ= {1,2,6} is the Poisson parameter; α= {1,2,6} is the Poisson parameter for the jump

pedestal and background

The signal is going to be convoluted with background processes. First we have to figure out what the background looks like in the absense of signal.

We take an integral (or sum) in a given time interval over which signal might arrive. If there is no signal then we end up adding a fluctuating baseline with some mean. This will have a Gaussian PDF

$P_Q(x) = \frac{1}{\sqrt{2\pi\sigma_0^2}} \mathrm{e}^{-\frac{(x-q_0)^2}{2\sigma_0^2}}$ where q_0 is the pedestal

There is another background process due to emission of real electrons from either the photocathode or one of dynodes due to thermal fluctuations. This will have an exponential PDF, but with some chance that there is no emission at all.

$\mathrm{P}_D(x) = (1-w)\delta(x) + w\theta(x)c_0 e^{-c_0 x}$

here w is the probability of emission, $\delta(x)$ is the dirac delta function to create a generalized distribution, and $\theta(x)$ is a step function, and c_0 is some constant.

background function

We first have to convolute P_Q and P_D to get the background only spectrum for

$B = Q + D$; We can do this explicitly or just write down the characteristic function.

$$\varphi_B(s) = \varphi_Q(s) \bullet \varphi_D(s) = e^{isq_0} e^{-\frac{1}{2}\sigma_0^2 s^2} \times \left((1-w) + w \frac{1}{1 - is / c_0} \right)$$

as long as the width of the pedestal is small with respect to the exponential $1/c_0$

$$P_B(x) = (1-w) \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(x-q_0)^2}{2\sigma_0^2}} + w\theta(x - q_0) c_0 e^{-c_0(x-q_0)}$$

Basically there is a pedestal with some width and a falling exponential background.

What if the pedestal width is too wide ? Then the exponential part of the background will get consumed in the width of the pedestal.

It is possible to obtain the full form.

Gaussian-modified-exponential

A normally distributed random number with an addition of an exponential random number is called an exponential-Gaussian or Gaussian-exponential.

The characteristic function is

$$\varphi_{EG}(s) = \frac{e^{isq_0} e^{-\frac{1}{2}\sigma_0^2 s^2}}{(1 - is / c_0)}$$

The PDF that corresponds to this is

$$P_{EG}(x) = \frac{c_0}{2} e^{\frac{c_0^2 \sigma_0^2}{2}} e^{-c_0(x-q_0)} Erfc\left[\frac{1}{\sqrt{2}}\left(c_0\sigma_0 - \frac{x-q_0}{\sigma_0}\right)\right]$$

Recall that c_0 is the constant for dark rate, σ_0 is the std. dev. of the pedestal and q_0 is the pedestal.

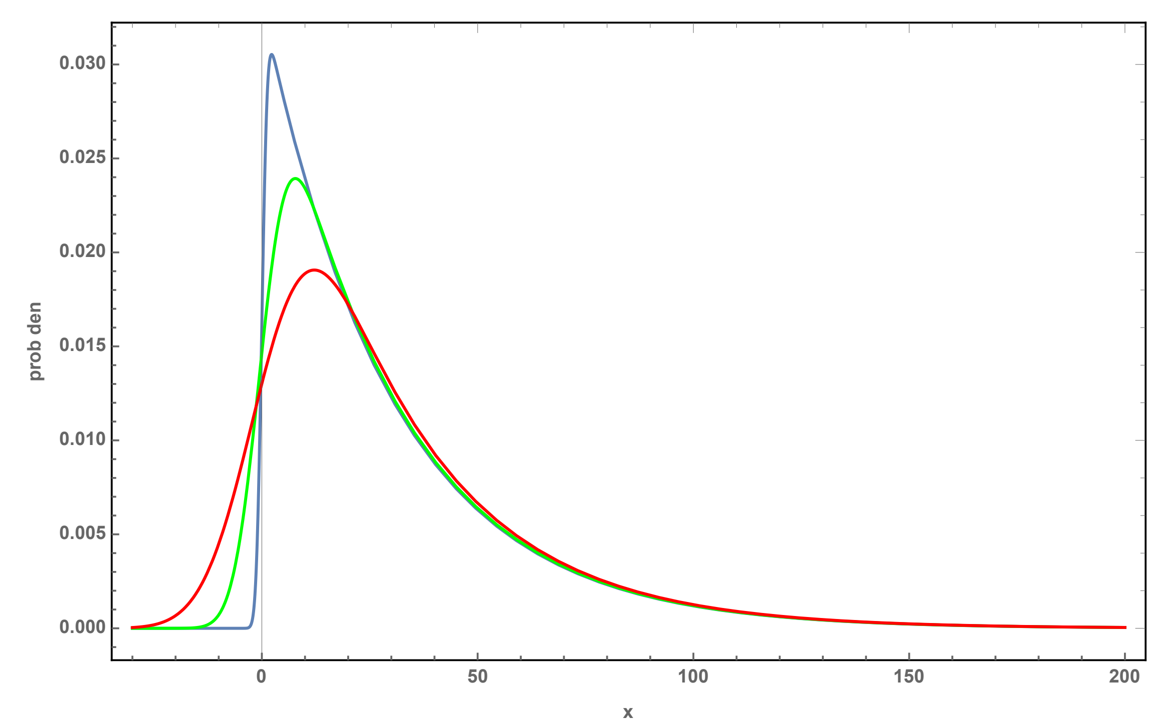
$Erfc[x]$ is the complement of the error fuction.

$$Erfc[x] = 1 - Erf[x] = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

When $c_0\sigma_0$ is small the $Erfc$ acts like a step function. Some care is needed in calculation in case of negative or very large arguments.

The Mean of the PDF is $(q_0 + 1/c_0)$

The Variance is $(\sigma_0^2 + 1/c_0^2)$



Couple of definitions

Let's define some PDFs to get a compact expression

The Normal PDF

$$N(x:\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\ \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The Exponential modified Normal PDF

$$E_N(x:\mu,\sigma^2,\lambda) = \frac{\lambda}{2} e^{\frac{\lambda^2\sigma^2}{2}} e^{-\lambda(x-\mu)} Erfc\left[\frac{1}{\sqrt{2}}\left(\lambda\sigma - \frac{x-\mu}{\sigma}\right)\right]$$

μ: mean of the Gaussian

σ^2: variance of the Gaussian

λ: exponential decay parameter

total response

To get the complete response we have to get the PDF for $Y = B+Z$ where

B is the background and Z is the signal. $B = D+Q$ as we calculated for the background.

and so

$$\varphi_Y(s) = \varphi_Z(s) \bullet \varphi_D(s) \bullet \varphi_Q(s)$$

$$\varphi_Y(s) = \left(\sum_{k=0}^{\infty} (e^{is\mu k} \bullet e^{-\frac{s^2\sigma^2 k}{2}}) \bullet \frac{e^{-\lambda}\lambda^k}{k!} \right) \times \left((1-w) + w\frac{1}{1-is/c_0} \right) \times e^{isq_0} e^{-\frac{1}{2}\sigma_0^2 s^2}$$

This is the full and complete expression for the PMT response assuming the gain is normally distributed. We will break this up in 6 pieces and analyze it for special conditions.

$$\varphi_Y(s) = \left(\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} (e^{is(\mu k + q_0)} \bullet e^{-\frac{s^2(\sigma^2 k + \sigma_0^2)}{2}}) \right) \times \left((1-w) + w\frac{1}{1-is/c_0} \right)$$

We break this up in three cases for p.e. count: $k=0$, $k=1$, and $k>1$

And additional two cases for $(1\text{-}w)$, without dark rate

and (w), with dark rate addition.

all terms broken out for the characteristic function. Check that when s=0 the sum adds to 1

	Terms	$(1-w)\times$	$w\times$
no signal	$e^{-\lambda}\times$	$e^{isq_0}e^{-\frac{s^2\sigma_0^2}{2}}$	$e^{isq_0}e^{-\frac{s^2\sigma_0^2}{2}}\times\frac{1}{1-is/c_0}$
single pe	$\lambda e^{-\lambda}\times$	$e^{is(\mu+q_0)}e^{-\frac{s^2(\sigma^2+\sigma_0^2)}{2}}$	$e^{is(\mu+q_0)}e^{-\frac{s^2(\sigma^2+\sigma_0^2)}{2}}\times\frac{1}{1-is/c_0}$
many pe	$\sum_{k=2}^{\infty}\frac{e^{-\lambda}\lambda^k}{k!}\times$	$e^{is(\mu k+q_0)}e^{-\frac{s^2(\sigma^2 k+\sigma_0^2)}{2}}$	$e^{is(\mu k+q_0)}e^{-\frac{s^2(\sigma^2 k+\sigma_0^2)}{2}}\frac{1}{1-is/c_0}$
		no dark current	*with dark current*

all these are to be added together. When transformed to PDF, each term will convert to a normal PDF or an exponential-normal PDF.

Approximation when pedestal is narrow, also set pedestal q0 = 0

	Terms	$(1-w)\times$	$w\times$
no signal	$e^{-\lambda}\times$	$e^{-\frac{s^2\sigma_0^2}{2}}$	$\dfrac{1}{1-is/c_0}$
single pe	$\lambda e^{-\lambda}\times$	$e^{is(\mu)}e^{-\frac{s^2(\sigma^2)}{2}}$	$e^{is(\mu)}e^{-\frac{s^2(\sigma^2)}{2}}\times\dfrac{1}{1-is/c_0}$
many pe	$\sum_{k=2}^{\infty}\dfrac{e^{-\lambda}\lambda^k}{k!}\times$	$e^{is(\mu k)}e^{-\frac{s^2(\sigma^2 k)}{2}}$	$e^{is(\mu k)}e^{-\frac{s^2(\sigma^2 k)}{2}}\dfrac{1}{1-is/c_0}$
		no dark current	*with dark current*

$$\sigma_0 << \sigma \text{ and } \sigma_0 << 1/c_0$$

$$\text{also set } q_0 = 0$$

when w=0 or c0 is very small

	$Terms$	$(1-w)\times$	$w\times$
no signal	$e^{-\lambda}\times$	$e^{-\frac{s^2\sigma_0^2}{2}}$	$\frac{1}{1-is/c_0}$
single pe	$\lambda e^{-\lambda}\times$	$e^{is(\mu)}e^{-\frac{s^2(\sigma^2)}{2}}$	$e^{is(\mu)}e^{-\frac{s^2(\sigma^2)}{2}}\times\frac{1}{1-is/c_0}$
many pe	$\sum_{k=2}^{\infty}\frac{e^{-\lambda}\lambda^k}{k!}\times$	$e^{is(\mu k)}e^{-\frac{s^2(\sigma^2 k)}{2}}$	$e^{is(\mu k)}e^{-\frac{s^2(\sigma^2 k)}{2}}\frac{1}{1-is/c_0}$
		no dark current	with dark current

$\sigma_0 << \sigma$ and $\sigma_0 << 1/c_0$

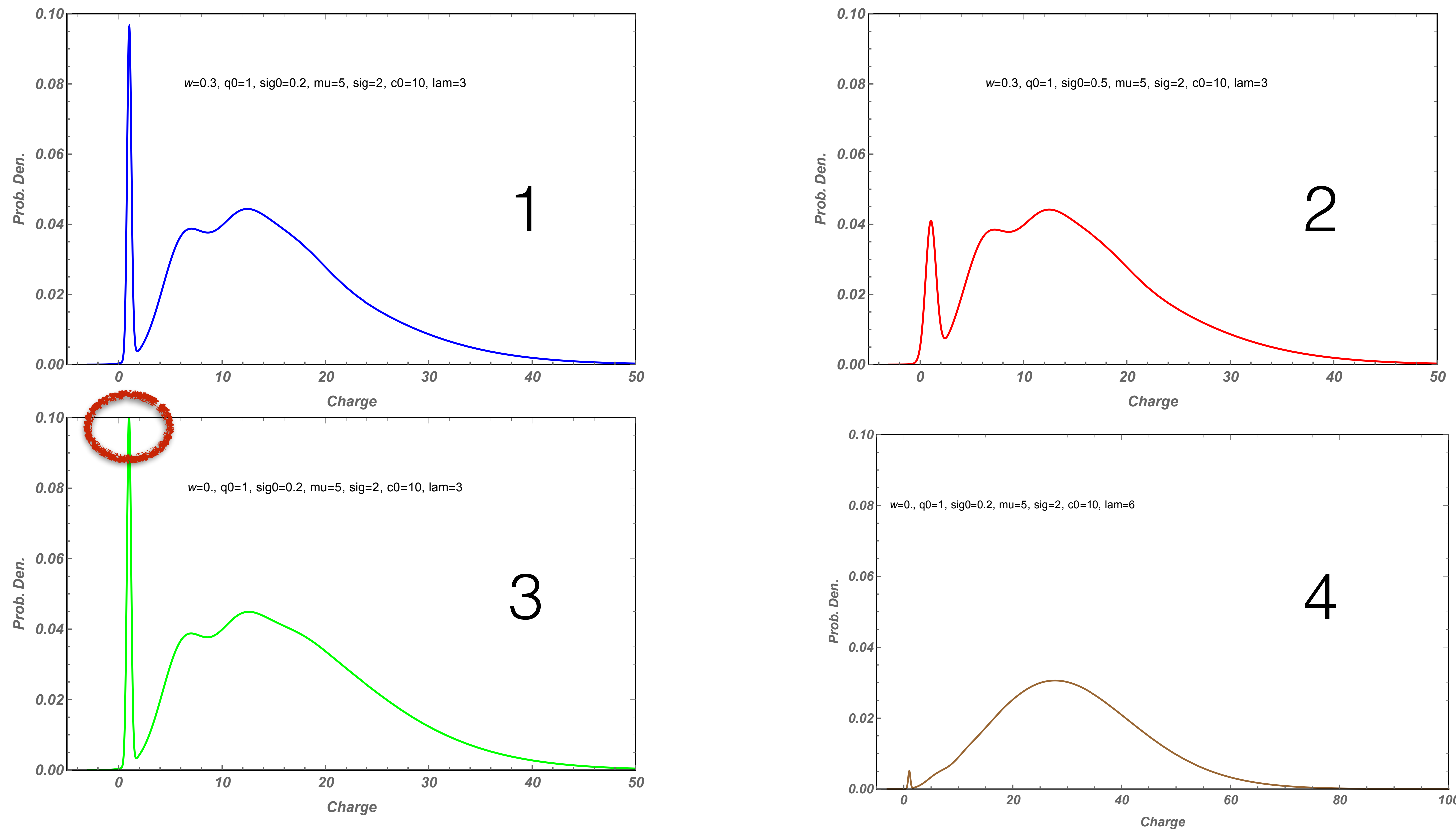
also set $q_0 = 0$

all terms broken out for the PDF in compact notation

	Terms	$(1-w)\times$	$w\times$
no signal	$e^{-\lambda}\times$	$N(x:q_0,\sigma_0^2)$	$E_N(x:q_0,\sigma_0^2,c_0)$
single pe	$\lambda e^{-\lambda}\times$	$N(x:\mu+q_0,\sigma^2+\sigma_0^2)$	$E_N(x:\mu+q_0,\sigma^2+\sigma_0^2,c_0)$
many pe	$\sum_{k=2}^{\infty}\frac{e^{-\lambda}\lambda^k}{k!}\times$	$N(x:\mu k+q_0,\sigma^2 k+\sigma_0^2)$	$E_N(x:\mu k+q_0,\sigma^2 k+\sigma_0^2,c_0)$
		no dark current	*with dark current*

all these are to be added together to get the full PDF. The sum is applied across the row.

plot some examples

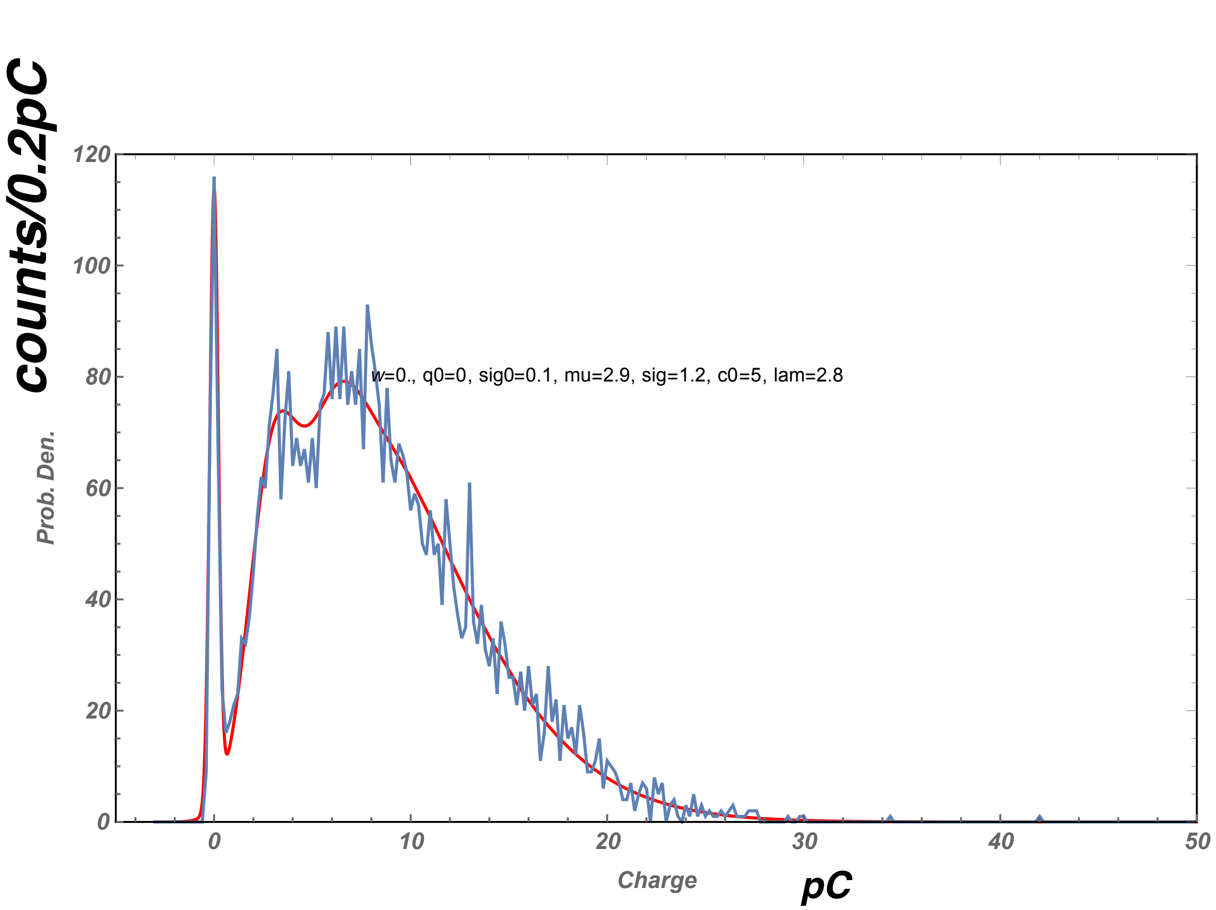


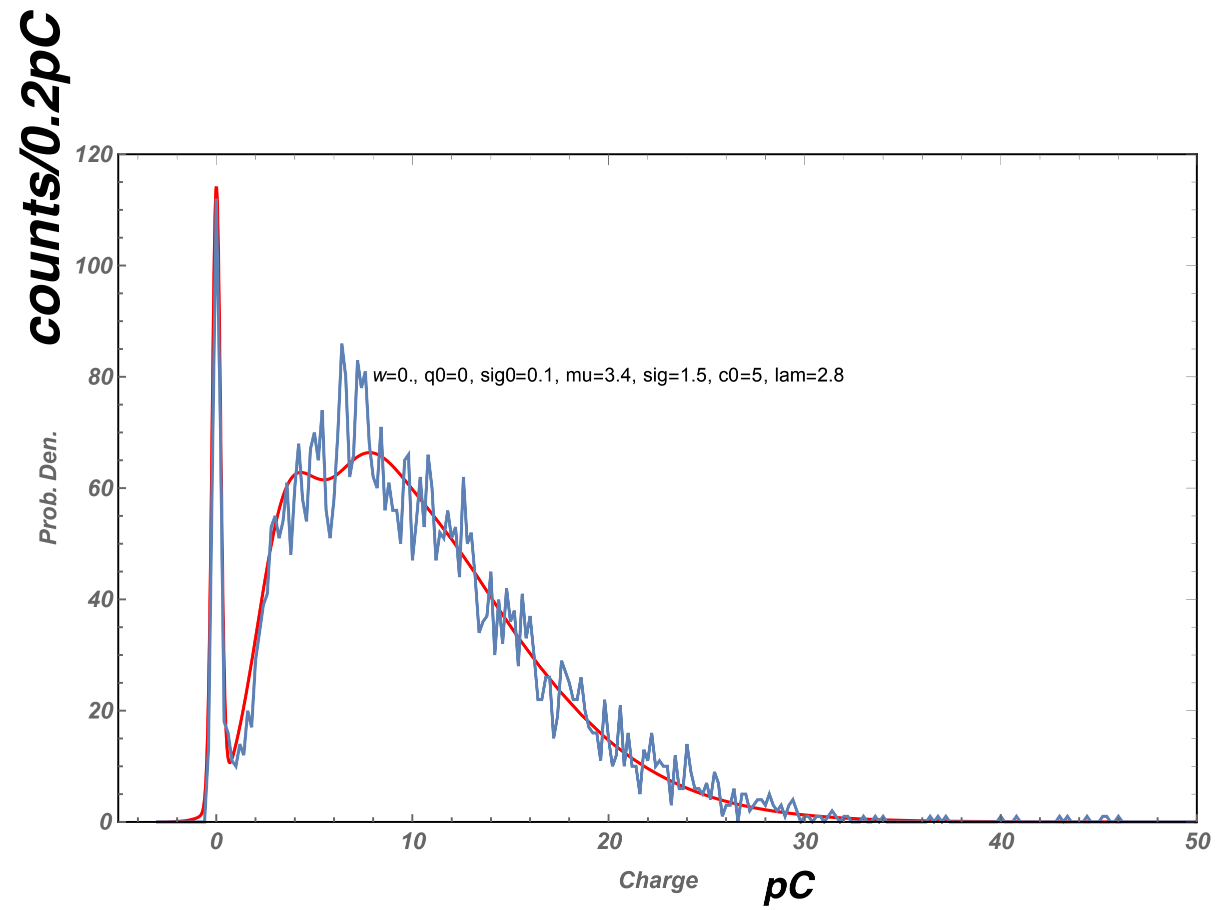
plot	λ	w	q_0	σ_0	μ	σ	c_0
1	3	0.3	1	0.2	5	2	10
2	3	0.3	1	0.5	5	2	10
3	3	0	1	0.2	5	2	10
4	6	0	1	0.2	5	2	10

with dark current, narrow/wide pedestal

no dark current, less/more p.e.

some data





- ***I have provided some data from a HPK R5912-mod 10 stage PMT. It has very low dark rate. An LED was flashed thru a fiber at the PMT.***
- ***Data was taken with a scope and so the pedestal noise is very low also.***
- ***There has been no selection of data. 5000 pulses were integrated in a fixed time interval and the LED pulse charge plotted with no cuts.***
- ***This PMT is B10-1 at 1430V (left), and 1460V (right)***
- ***The red curve is not a fit, I just guessed at the parameters.***
- ***Homework: fit the 6 spectra I have provided.***

conclusion.

- ***We derived the full expression for the charge spectrum from a typical photo-multiplier.***
- ***The expression has parameters for the pedestal, width of the pedestal, the dark current, and the signal.***
- ***The method for deriving the expression is very general, and can be applied to any detector system with appropriate changes.***
- ***The expression can be used for a full fit to an experimental spectrum. It is important to know the stable conditions under which data was obtained.***